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Bifurcation in a time-optimal problem for a second-order non-linear system☆

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ABSTRACT

A time-optimal problem for a second-order non-linear system with one degree of freedom is considered. The system describes the dynamics of an inertial object under the action of a control force of limited modulus which appears linearly and a perturbing force which is periodic in coordinate. The terminal set represents points on the abscissa of the phase plane, and the distance between two neighbouring points is equal to the period of the perturbing force. An estimate is obtained for the amplitude of the control for which the control has the simplest structure: the number of switchings is not greater than one.

It follows from the maximum principle[1] that a time-optimal control takes limiting values. For the optimal trajectory, the switching of a control from one limiting value to another can occur once or several times. The number of switchings depends on the maximum admissible value of the control, which is conveniently normalized to the maximum value of the perturbing force. It was found that, if this ratio is greater than a certain value, which we shall call the principal bifurcation value, then there will be no more than one switching for each optimal trajectory.

In the special cases when a non-linear function defines the perturbing force, the system considered describes the plane oscillations of a single-section pendulum or the rotation of a satellite around its own axis in the circular orbit plane.

Different feedback control methods, which transfer a pendulum into a lower stable or an upper unstable equilibrium position in the case of a limited modulus of the control moment, have been proposed in a set of publications. The problem of the optimal synthesis in the case of large control amplitudes (when the maximum admissible value of the control has a modulus which is greater than the maximum moment of the gravitational force) has been considered in several of them but the solution turned out to be incomplete and the threshold restriction on the modulus of the control moment, for which there are no trajectories with more than one control switching in the optimal synthesis pattern, was not indicated or incorrectly indicated. As a result of this, only the numerically obtained parts of the phase pattern close to the terminal set were shown and the main features of the optimal synthesis in the case of large angular velocities of rotation, that is, far from the terminal state, were not investigated, (see, for example, see Refs 2 and 3, where the terminal set is the lower equilibrium position). The size of this neighbourhood was established[3] by means of an additional constraint imposed on the time of motion along optimal trajectories which was chosen depending on the modulus of the admissible control moment.

A more complete solution of the optimal synthesis problem for the build up and decay of the oscillations of a pendulum was presented in Refs 4–7. The solution of both problems is based on the maximum principle and involves an analytical investigation in conjunction with numerical calculations. As a result, switching curves (SCs) and dispersion curves (DCs) (which, according to the existing terminology[8], restrict the domains in phase space corresponding to ratious values of a discontinuous optimal control) are constructed for various values of the maximum admissible control moment. The structure of these curves as well as the fields of the optimal trajectories have been investigated in detail. This structure depends very much on the magnitude of the control moment. In particular, numerical calculations reveal the mechanism of the formation of numerous discontinuities in the SCs (in the problem of the quenching of oscillations) on changing from the case of large moments, when these curves are smooth, to the case of small moments, corresponding to SCs with discontinuities.

In the problem of the quenching of oscillations, singularities arise in the optimal synthesis pattern far from the terminal points. The bifurcation process was therefore studied in detail,[7] a principal bifurcation value $\approx$1.04 was found and its analytical upper limit was obtained. An extension of this estimate to the case of a second order system of general form is given in this paper.

In the problem of the build up of oscillations, no singularities arise in the optimal synthesis pattern far from the terminal points. Two bifurcation values: $\approx$0.80 and $\approx$0.44 were indicated[4,5] with an accuracy of up to hundredths. In the case of the first of them (the principal

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value) optimal trajectories appear with two switchings, and, in the second case, with three switchings. It is sufficient to analyse the optimal synthesis pattern visually in order to locate the bifurcations.

A controlled mechanical system in the form of a pendulum with the suspension point on the axis of a wheel which can roll along a horizontal without slipping (an elliptic pendulum) has been investigated;[9] the control moment applied to the wheel is of limited modulus. The time-optimal control for the quenching of the oscillations of the pendulum was constructed in the phase cylinder. An algorithm was given for calculating a preset control which transfers the system from the lower equilibrium position into the upper unstable equilibrium position with quenching of the velocity of the suspension point at the end of the motion. The equation of motion of the elliptic pendulum in Ref. 9 is distinguished from the system considered in this paper by a factor in front of the control which depends on a coordinate.

The problem of the optimal steering of an artificial Earth satellite into a gravitational-stable position in the case of motion in a circular orbit was considered in Ref. 10. The solution is based on the results in Ref. 16, where the cylindricity of the phase space arising in satellite control problems was additionally taken into account. It was proved that the so-called FLAG domain with an index 1, within which optimal trajectories with two switchings can be started (see Section 2), is located to the right of the critical partition in the phase cylinder in the case of large control moments and, therefore, cannot be taken into consideration when constructing the optimal synthesis.

A numerical and qualitative solution of a program problem involving the rotation of a pendulum, initially in the lower stable position, by 360° around the suspension point was given in Ref. 11. This formulation of the problem also corresponds to the controlled rotation of a satellite in the circular orbit plane by 180° from one position into another, both of which correspond to stable orbital motion of the satellite. The dependences of the response time and the number of switchings on the moment were constructed. As a result, the sequence of the first five bifurcation values was found numerically: 0.80, 0.44, 0.30, 0.23.

A description of the system is given in Section 1 and corresponding examples are presented. The time-optimal problem is formulated in Section 2 and the known properties of the optimal solution are indicated. An upper analytical estimate of the principal bifurcation value is found in a general form in Section 3.

1. Description of the system

The controlled non-linear dynamical system

$$J\ddot{\varphi} + \varepsilon(\varphi) = M \tag{1.1}$$

is considered, where $J > 0$ is a constant inertial characteristic of the system.

We shall say that a scalar control $M(t)$ is admissible if it satisfies the following two conditions:

1) $M(t)$ is a piecewise-continuous function of time,
2) for all $t \in [0, +\infty]$

$$|M(t)| \le a, \quad a = \text{const} \tag{1.2}$$

The function $\varepsilon(\varphi)$ satisfies the three conditions:

1) it is periodic with a period of 2θ,
2) it is continuously differentiable in the interval $(-\infty, +\infty)$,
3) its mean value over a period is equal to zero:

$$\int_0^{2\theta} \varepsilon(\varphi) d\varphi = 0 \tag{1.3}$$

We will denote the maxima of the moduli of the function $\varepsilon(\varphi)$ and its derivative $d\varepsilon/d\varphi$ by b and c, that is,

$$b = \max_{\varphi} |\varepsilon(\varphi)|, \quad c = \max_{\varphi} \left|\frac{d\varepsilon}{d\varphi}(\varphi)\right| \tag{1.4}$$

The initial states of system (1.1) at the instant $t = 0$ are arbitrary:

$$\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \dot{\varphi}_0 \tag{1.5}$$

and the terminal conditions at some instant $t = T$ ($T > 0$) are given in the form

$$\varphi(T) = \varphi_T + 2\theta n, \quad \dot{\varphi}(T) = 0 \tag{1.6}$$

where $\varphi_T \in [0, 2\theta]$ is a fixed quantity and n is an arbitrary integer. The instant T of entry into the terminal set (1.6) is not fixed and depends on the choice of the control $M(t)$.

Example 1. In the case of a pendulum control, J and M in the equation of motion (1.1) are the moment of inertia of the pendulum and the bounded control moment with respect to the suspension point, and the function $\varepsilon(\varphi)$ has the form

$$\varepsilon(\varphi) = mgl \sin\varphi$$

where m and l are the mass and length of the pendulum, and g is the acceleration due to gravity. If $\varphi_T = 0$ in the terminal conditions (1.6), then we are dealing with the problem of bringing the pendulum into the lower stable equilibrium position, and, if $\varphi_T = \pi$, into the upper unstable equilibrium position. In the general case, φ_T can be an arbitrary number from the interval $[0, 2\pi)$. The half-period in this example is equal to π.

Example 2. The rotation of a satellite around its own axis in the circular orbit plane under the action of an external or internal control moment is described by the equation[10,12,13]

$$\ddot{\varphi} + 3\omega^2 \frac{A-C}{B} \sin\varphi = 2M, \quad \varphi = 2\nu \tag{1.7}$$

Here, ω is the angular velocity of orbital motion of the satellite, A, B and C are its principal central moments of inertia, ν is the angle between the C axis of the satellite and its radius vector, and M is the control moment. By assumption, $A > C$ and the B axis of the satellite is perpendicular to the orbital plane. Note that the control moment can be both an external moment, produced by a jet engine, as well as an internal moment which arises due to the presence of rotating internal masses (the case of a satellite-gyrostat[12]). If $M = 0$, then physically different stable equilibrium positions of the satellite exist:

$$\nu = 2\pi n, \quad (\varphi = 4\pi n) \tag{1.8}$$

$$\nu = \pi + 2\pi n, \quad (\varphi = 2\pi + 4\pi n) \tag{1.9}$$

They correspond to orientations of the satellite along the radius vector (see (1.8)) or opposed to it (see (1.9)).

Suppose it is required to transfer the satellite into one of the above-mentioned positions under the condition $\dot{\nu} = 0\,(\dot{\varphi} = 0)$. Comparing the terminal conditions 91.6) with relations (1.8) and (1.9), we determine the corresponding values of φ_T: $\varphi_T = 0$ in the first case and $\varphi_T = 2\pi$ in the second case. In the general case, $\varphi_T \in [0, 4\pi)$ is an arbitrary number. The half-period in this case is equal to 2π.

2. The time-optimal problem

We will introduce the dimensionless variables

$$\varphi' = \frac{c(\varphi - \varphi_T)}{b}, \quad \theta' = \frac{c\theta}{b}, \quad t' = \left(\frac{c}{J}\right)^{1/2} t, \quad T' = \left(\frac{c}{J}\right)^{1/2} T$$

$$u = \frac{M}{a}, \quad k = \frac{a}{b}, \quad \varepsilon'(\varphi') \equiv \frac{1}{b}\varepsilon\left(\frac{b\varphi'}{c} + \varphi_T\right)$$

and write Eq. (1.1) in the following form (henceforth we will use dots to denote derivatives with respect to the dimensionless time t' and we will omit the prime on new variables)

$$\ddot{\varphi} + \varepsilon(\varphi) = ku \tag{2.1}$$

Constraints (1.2) and (1.4) can be represented as

$$|u(t)| \le 1 \tag{2.2}$$

$$|\varepsilon(\varphi)| \le 1, \quad \left|\frac{d\varepsilon}{d\varphi}(\varphi)\right| \le 1 \tag{2.3}$$

The form of the initial conditions (1.5) does not change, but terminal conditions (1.6) are transformed in the following manner

$$\varphi(T) = 2\theta n, \quad \dot{\varphi}(T) = 0 \tag{2.4}$$

We will first formulate the time-optimal response problem while not paying much attention to the method of solving it.

Problem 1. For a given parameter $k > 0$, it is required to construct a feedback control $u(\varphi, \dot{\varphi})$ which satisfies constraint (2.2) and transfers system (2.1) from an arbitrary initial state into the terminal state (2.4) in the minimum time T.

Problem 1 can be solved, for example, using the relations of the maximum principle.[1] It follows from the maximum principle that the optimal control takes the values $u(t) = \pm 1$. The sign of the control is equal to the sign of the conjugate variable corresponding to $\dot{\varphi}$. In order to obtain the optimal control in the form of a synthesis $u(\varphi, \dot{\varphi})$, it is sufficient to find the switching curves (SCs) and the dispersion curves (DCs) in the $\varphi\dot{\varphi}$ plane, which bound the domain where $u = +1$ and $u = -1$. There are no singular controls in Problem 1.[2]

Note that the SCs consist of points at which the control u changes its sign in the motion along an optimal trajectory and the DCs are formed by the points where the optimal control can be equal to either +1 or -1, and the two optimal trajectories, starting from each of these points, reach the terminal state (2.4) (in the case of the same or different n) after the same time. A non-uniqueness of the optimal trajectories therefore holds.

It is well known that the invariance of Eq. (2.1) and the terminal set (2.4) with respect to the shift transformation $\varphi \to \varphi + 2\theta$ leads to a structure (cylindricity) of the synthesis pattern which is periodic in coordinate. The cylindricity of the phase space also causes other specific features of the synthesis. The main one of these is the existence of a DC in the cylinder, from each point of which two optimal trajectories with the same time of motion emerge and lead into two adjacent terminal points (2.4) corresponding to different n.

Obviously, only a numerical construction of the optimal synthesis in the phase cylinder is possible in Problem 1 in the general case (see Refs 2, 10, and 14) in spite of the fact that the system of differential equations of the maximum principle here can be successfully integrated in quadratures[14].

In the case of an infinitely large control action when $k \to \infty$, the non-linear term in equation of motion (2.1) can be omitted. The synthesis pattern in this case consists of parabolic SCs passing through the terminal points and DCs which are arranged between them. The equation

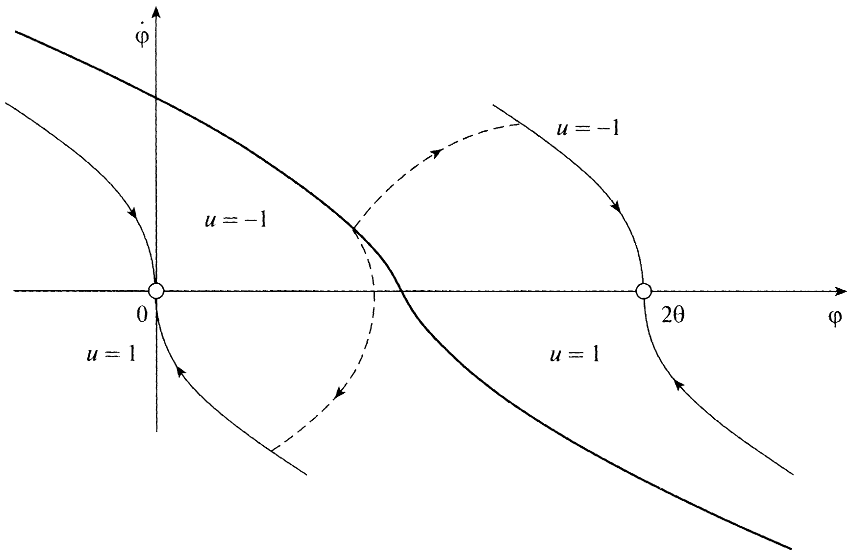

Fig. 1.

of the DCs in the case of an infinitely large control action can be obtained in explicit form.[15] The optimal trajectories do not have more than one switching. Two neighbouring terminal points in the phase space, the SC passing through them (the thin solid lines), the DC located between them (the thick solid line) and some optimal trajectories (the broken lines) are shown in Fig. 1. The arrows indicate the direction of motion along the optimal trajectories.

In the case of large control moments when $k > 1$, optimal trajectories with two switchings of the control can also exist.[16] Note that an analysis of the problem of the control of a second-order system of the form of (2.1) has been given earlier[16] but the time-optimal transfer of any point of the phase plane to the origin of coordinates was considered and the cylindricity of the phase space was not taken into account; the control was assumed to be sufficiently large. It has been shown[16] that, besides the central SCs, a further denumerable set of SCs, which depart to infinity, exists. Each of these additional SC continuously connect with another singular SC, «an indifferent curve» which also departs to infinity. Two optimal trajectories (with the same time of motion) originate from the points lying on an indifferent curve; one of these has a single switching and the other has two. Each indifferent curve has a common point with a corresponding SC and forms the boundary of the simply connected FLAG domain in which the optimal trajectories with two switchings originate (the term FLAG was introduced by Beletskii[10] and is formed from the first letters of the names of the authors of Ref. 16). The above mentioned domains are characterized by the indices 0, 1, 2, ... The formation of new FLAG domains in the phase plane occurs in a bifurcation manner at infinity in proportion to the reduction in the control possibilities. However, even in the case of k values which are as large as desired, infinitely many FLAG domains with a large index exist in the phase plane, that is, optimal trajectories with two switchings always exist.

In the case of phase space cylindricity, optimal trajectories with two (or more) switchings only appear with the constraints k: $k \le k^*$, where k^* is the principal bifurcation value. The constant k^* can be both greater than and less than unity.

3. Basic results

We will now formulate the results obtained in the analytical investigation of the time-optimal problem in the form of a theorem.

Theorem. A number $\tilde{k}$ exists such that, when $k \ge \tilde{k}$, the control in Problem 1 does not have more than one switching.

Proof. Trajectories with two switchings. It is assumed that, in system (2.1), the condition

$$k > 1 \tag{3.1}$$

is satisfied, corresponding to the case of a large control force in the initial system (1.1), that is, when the possibilities of the control force $M(t)$ exceed the possibilities of the non-control force $\varepsilon(\varphi)$. Constraint (3.1) is imposed in order to ensure a monotonic change in the phase velocity $\dot{\varphi}(t)$ in the case of a constant control $u = \pm 1$. It should be noted that constraint (3.1) is unimportant, since the estimate of the principal bifurcation value $\tilde{k}$ obtained below was found to be greater than unity. □

We will denote the band in phase space bounded by the trajectories of the system arriving at the points (0,0) and (2θ, 0) in the case of a constant control $\dot{u} = \pm 1$ by S.

We will now consider the auxiliary trajectory (see Fig. 2) which originates at a certain point (φ_0, $\dot{\varphi}_0$) within the band S, arrives at the point (0,0) and has two switchings of the control at the points (φ_1, $\dot{\varphi}_1$) and (φ_3, $\dot{\varphi}_3$).

If this trajectory is optimal, then it entirely belongs to the band S and intersects the abscissa at a point (φ_2, 0) at some instant between the first and second switching of the control. This assertion follows from Remarks 1°-3°.

Remarks. 1° The points (φ_1, $\dot{\varphi}_1$) and (φ_3, $\dot{\varphi}_3$) lie on different sides of the abscissa. This fact follows from a theorem[2] on the alternation of the roots of the function $\dot{\varphi}$ and the function which is conjugate to it, the sign of which determines the sign of the optimal control.

2°. The points (φ_1, $\dot{\varphi}_1$) and (φ_3, $\dot{\varphi}_3$) lie to the left of trajectories with a constants control $u = \pm 1$ leading to the point (2θ, 0) since, otherwise, it would be more advantageous to move with a single switching to the point (2θ,0) and not (0,0).

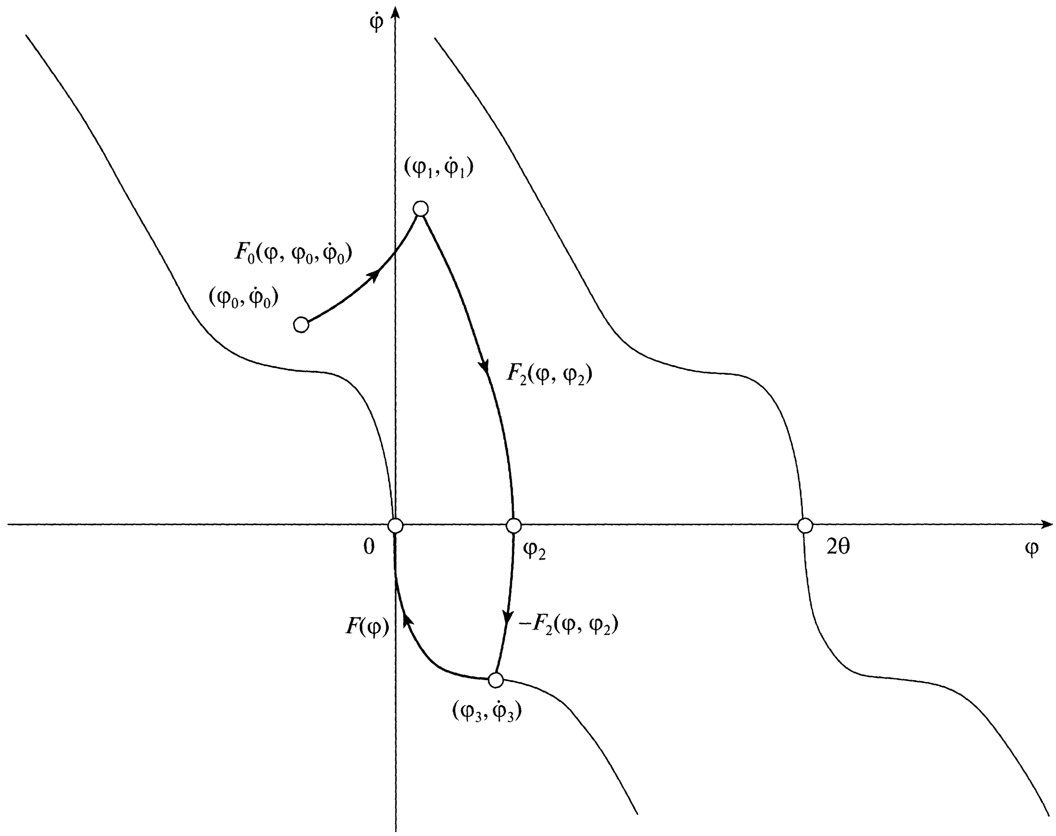

Fig. 2.

3°. Motions with two switchings from a selected point $(\varphi_0, \dot\varphi_0)$ other terminal points (2.4) (when $n>0$) are slower and automatically non-optimal.

The time of motion along a trajectory depends on three parameters: $T = T(\varphi_2, \varphi_0, \dot\varphi_0)$. We will obtain a specific expression for this function, assuming, without loss of generality (see Remark 4° below), that

$$\dot\varphi_0 > 0 \tag{3.2}$$

To do this, we introduce the notation F, F_0 and F_2, for the functions specifying the velocity of the system $\dot\varphi$ in the segments of a trajectory with a constant control (see Fig. 2).

Using the expression for the first integral

$$\frac{\dot\varphi^2}{2} + f(\varphi) = ku\varphi + C, \quad u = \text{const} \tag{3.3}$$

$$f(\varphi) = \int_0^\varphi \varepsilon(\sigma)d\sigma + 1 \tag{3.4}$$

we obtain the form of the functions F, F_0 and F_2 and we also determine the coordinates φ_1, φ_3 of the control switching points as functions of φ_2, φ_0, $\dot\varphi_0$ (φ_3 (φ_2) and $\varphi_1(\varphi_2, \varphi_0, \dot\varphi_0)$):

$$\begin{gathered} F(\varphi): \quad u = 1, \quad C = 0 \\ F(\varphi) = -\sqrt{2k\varphi - 2f(\varphi)} \end{gathered} \tag{3.5}$$

$$\begin{gathered} F_2(\varphi, \varphi_2): \quad u = -1, \quad C = f(\varphi_2) + k\varphi_2 \\ F_2(\varphi, \varphi_2) = \sqrt{-2k(\varphi - \varphi_2) - 2[f(\varphi) - f(\varphi_2)]} \end{gathered} \tag{3.6}$$

$$\begin{gathered} F_0(\varphi, \varphi_0, \dot\varphi_0): \quad u = 1, \quad C = \frac{\dot\varphi_0^2}{2} + f(\varphi_0) - k\varphi_0 \\ F_0(\varphi, \varphi_0, \dot\varphi_0) = \sqrt{2k(\varphi_3 - \varphi_0) - 2[f(\varphi) - f(\varphi_0)] + \dot\varphi_0^2} \end{gathered} \tag{3.7}$$

We express φ_3 in terms of φ_2. At the point $(\varphi_3, \dot\varphi_3)$, we have the equality

$$F^2(\varphi_3) = F_2^2(\varphi_3, \varphi_2) \tag{3.8}$$

from which, using expressions (3.5) and (3.6), we obtain

$$\varphi_3(\varphi_2)=\frac{\varphi_2}{2}+\frac{f(\varphi_2)}{2k} \tag{3.9}$$

We express φ_1 in terms of φ_2, φ_0 and $\dot\varphi_0$. At the point $(\varphi_1, \dot\varphi_1)$ we have the equality

$$F_0^2(\varphi_1,\varphi_0,\dot\varphi_0)=F_2^2(\varphi_1,\varphi_2) \tag{3.10}$$

from which, using expressions (3.6), (3.9) and (3.7), we obtain

$$\varphi_1(\varphi_2,\varphi_0,\dot\varphi_0)=\varphi_3(\varphi_2)+\psi(\varphi_0,\dot\varphi_0) \tag{3.11}$$

The expression for the function $\psi(\varphi_0, \dot\varphi_0)$ is not given (see Remark 5° below).

We next use the well-known formula for finding the time of motion from a point $x = x_a$ to a point $x = x_b$ with a velocity $dx/dt = \upsilon(x)$:

$$T=\int_{x_a}^{x_b}\frac{dx}{\upsilon(x)}$$

As a result, we find the time of motion

$$T=\int_{\varphi_0}^{\varphi_1(\varphi_2,\varphi_0,\dot\varphi_0)}\frac{d\varphi}{F_0(\varphi,\varphi_0,\dot\varphi_0)}+\int_{\varphi_1(\varphi_2,\varphi_0,\dot\varphi_0)}^{\varphi_2}\frac{d\varphi}{F_2(\varphi,\varphi_2)}+\int_{\varphi_3(\varphi_2)}^{\varphi_2}\frac{d\varphi}{F_2(\varphi,\varphi_2)}-\int_0^{\varphi_3(\varphi_2)}\frac{d\varphi}{F(\varphi)} \tag{3.12}$$

The necessary condition for optimality. If the trajectory considered is optimal, then the derivative $\partial T/\partial\varphi_2$ must vanish in the case of the values of φ_2, φ_0, $\dot\varphi_0$ being considered so that the equality

$$\frac{\partial T}{\partial\varphi_2}=0 \tag{3.13}$$

holds.

We will now obtain an expression for the derivative $\partial T/\partial\varphi_2$. Denoting the corresponding terms in formula (3.12) by T_1, T_2, T_3, T_4, we find the partial derivatives of these quantities with respect to the variable φ_2. At the same time, we take account of the fact that, according to equality (3.11),

$$\frac{\partial[\varphi_1(\varphi_2,\varphi_0,\dot\varphi_0)]}{\partial\varphi_2}=\frac{d[\varphi_3(\varphi_2)]}{d\varphi_2}$$

We have

$$\frac{\partial T_1}{\partial\varphi_2}=\frac{d\varphi_3}{d\varphi_2}\frac{1}{F_0(\varphi_1,\varphi_0,\dot\varphi_0)},\quad \frac{\partial T_4}{\partial\varphi_2}=-\frac{d\varphi_3}{d\varphi_2}\frac{1}{F(\varphi_3)} \tag{3.14}$$

Making the intermediate replacement $\varphi - \varphi_2 = y$ in the second and third integrals in formula (3.12), we differentiate them with respect to φ_2 and then change in the resulting expressions to the initial integration variable φ:

$$\frac{\partial T_2}{\partial\varphi_2}=\left(1-\frac{d\varphi_3}{d\varphi_2}\right)\frac{1}{F_2(\varphi_1,\varphi_2)}+I(\varphi_1,\varphi_2) \tag{3.15}$$

$$\frac{\partial T_3}{\partial\varphi_2}=\left(1-\frac{d\varphi_3}{d\varphi_2}\right)\frac{1}{F_2(\varphi_3,\varphi_2)}+I(\varphi_3,\varphi_2)$$
$$I(\varphi_1,\varphi_2)=\int_{\varphi_1}^{\varphi_2}\frac{[\varepsilon(\varphi)-\varepsilon(\varphi_2)]d\varphi}{F_2^3(\varphi,\varphi_2)} \tag{3.16}$$

The integrals in the expressions which have been obtained converge.

Summing the quantities (3.14)–(3.16) which have been found and taking account of the fact that

$$F_0(\varphi_1,\varphi_0,\dot\varphi_0)=F_2(\varphi_1,\varphi_2),\quad -F_2(\varphi_3,\varphi_2)=F(\varphi_3)$$

we obtain the required expression for the derivative

$$\frac{\partial T}{\partial\varphi_2}=\frac{1}{F_2(\varphi_1,\varphi_2)}+\frac{1}{F_2(\varphi_3(\varphi_2),\varphi_2)}+I(\varphi_1,\varphi_2)+I(\varphi_3(\varphi_2),\varphi_2) \tag{3.17}$$

Expression (3.17) is represented in the form of a function of only two parameters φ_1 and φ_2 and does not depend directly on the position of the initial point $(\varphi_0, \dot\varphi_0)$ in the phase cylinder. Only the value of φ_1 was related to the coordinates of this point using formula (3.11).

Remark. 4°. If assumption (3.2) were to be violated, the expression for the time of motion $T(\varphi_2, \varphi_0, \dot\varphi_0)$ in (3.12) would be changed. However, in this case, expressions (3.17) for the derivative $\partial T/\partial\varphi_2$ would not be changed. Assumption (3.2) is therefore unimportant.

5°. The form of the function $\psi(\varphi_0, \dot{\varphi}_0)$ in formula (3.11) has no effect on expression (3.17) for the derivative $\partial T/\partial\varphi_2$ and is therefore unimportant.

Estimate of the principal bifurcation value. We will now indicate those values of k for which expression (3.17) for $\partial T/\partial\varphi_2$ is strictly positive for any admissible φ_1 and φ_2 or, in other words, Eq. (3.13) does not have a solution when

$$-\infty < \varphi_1 \le \varphi_2, \quad 0 < \varphi_2 < 2\theta$$

In this case, it is guaranteed that there are no optimal trajectories with two control switchings in the domain S. Hence, an upper analytical estimate of the principal bifurcation value k will then be obtained. Here, the relations

$$|f(\alpha) - f(\beta)| \le |\alpha - \beta|, \quad |\varepsilon(\alpha) - \varepsilon(\beta)| \le |\alpha - \beta|, \quad |\varepsilon(\alpha) - \varepsilon(\beta)| \le 2$$

will be used, which are satisfied in the case of arbitrary α and β and follow from relations (2.3) and (3.4).

The first term in expression (3.17) is positive. We will estimate the second term:

$$\frac{1}{F_2(\varphi_3, \varphi_2)} = \frac{1}{\sqrt{-2k(\varphi_3 - \varphi_2) - 2|f(\varphi_3) - f(\varphi_2)|}} \ge \frac{1}{\sqrt{2(k+1)(\varphi_2 - \varphi_3)}} \ge \frac{1}{2\sqrt{\theta(k+1)}} \tag{3.18}$$

We will now estimate the third term. We choose a certain auxiliary constant η (its value will be indicated below) and, putting $\eta > 0$, we obtain

$$\begin{aligned} |I(\varphi_1, \varphi_2)| &\le \int_{-\infty}^{\varphi_2} \frac{|\varepsilon(\varphi) - \varepsilon(\varphi_2)| d\varphi}{[-2k(\varphi - \varphi_2) - 2|f(\varphi) - f(\varphi_2)|]^{3/2}} \le \frac{1}{[2(k-1)]^{3/2}} \left(\int_{-\infty}^{\varphi_2 - \eta} \frac{2d\varphi}{(\varphi_2 - \varphi)^{3/2}} + \right. \\ &\left. + \int_{\varphi_2 - \eta}^{\varphi_2} \frac{d\varphi}{(\varphi_2 - \varphi)^{1/2}} \right) = \frac{1}{(k-1)^{3/2}} \left\{ \sqrt{\frac{2}{\eta}} + \sqrt{\frac{\eta}{2}} \right\} \le \frac{2}{(k-1)^{3/2}} \end{aligned} \tag{3.19}$$

The last component in the chain of inequalities (3.19) is written taking account of the fact that a minimum of the expression in the braces is attained if $\eta = 2$ is chosen. Estimating the fourth term in a similar manner, we obtain

$$|I(\varphi_3, \varphi_2)| \le \frac{1}{[2(k-1)]^{3/2}} \int_0^{\varphi_2} \frac{d\varphi}{(\varphi_2 - \varphi)^{1/2}} \le \frac{\sqrt{\theta}}{(k-1)^{3/2}} \tag{3.20}$$

We require that the lower bound (3.18) should exceed the sum of the upper bounds (3.19) and (3.20). In this case, we arrive at the cubic inequality

$$\delta(k) \ge 0 \tag{3.21}$$

which limits the parameter k. The notation

$$\delta(k) = (k-1)^3 - C_1(k+1), \quad C_1 = 4\theta\left(2 + \sqrt{\theta}\right)^2 \tag{3.22}$$

has been introduced here

The corresponding cubic equation, defined by the equality sign in relation (3.21), has a unique root $k = \tilde{k}$ in the interval $[1, +\infty)$.

Actually, the function $\delta(k)$ is negative at its left end and positive at its right end.

$$\delta(1) = -2C_1 < 0, \quad \lim_{k \to +\infty} \delta(k) = +\infty$$

The derivative $d\delta/dk$ is negative when $k = 1$ but, it increases and becomes positive as k increases.

The solution of inequality (3.21) can therefore be written in the explicit form

$$\begin{aligned} &k \ge \tilde{k} \\ &\tilde{k} = \frac{C_2^{1/3}}{3} + \frac{C_1}{C_2^{1/3}} + 1, \quad C_2 = 27C_1 + 3\sqrt{-3C_1^3 + 81C_1^2} \quad \text{When} \quad C_1 \le 27 \\ &\tilde{k} = 2\sqrt{\frac{C_1}{3}} \cos\frac{\operatorname{arctg}\sqrt{C_1/27 - 1}}{3} + 1 \quad \text{When} \quad C_1 \ge 27 \end{aligned} \tag{3.23}$$

The expression for $\tilde{k}$ is determined using Cardano's formula. The quantity C_1 is related to θ as given by the second equality of (3.22).

The expression obtained for $\tilde{k}$ is the required estimate of the principal bifurcation value k and depends solely on the half-period θ.

Note that finite segments with two switchings always exist in optimal trajectories with three or more switchings. The proof therefore also covers this case.

Constraint (1.3), imposed earlier on the mean value of the function $\varepsilon(\varphi)$, can be rejected. In this case, Problem 1 does not always have a solution for small k. However, when $k > 1$, it is guaranteed that a solution of Problem 1 exists. At the same time, the formulation of the theorem and estimate (3.23) remain in force.

4. Conclusion

The results can be used to investigate of problems of the control of certain mechanical systems, including the control of a pendulum (see Ref. 7) and problems of the re-orientation of a satellite in a circular orbit or of steering it into a state with a given orientation. The discovery or more precise estimation of other bifurcation values k corresponding to the appearance of optimal trajectories with a specific number of switchings: with two, three or more, is also of interest.

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